Analytic Approach to Perturbative QCD

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Abstract

The two-loop invariant (running) coupling of QCD is written in terms of the Lambert W function. The analyticity structure of the coupling in the complex Q^2-plane is established. The corresponding analytic coupling is reconstructed via a dispersion relation. We also consider some other approximations to the QCD β-function, when the corresponding couplings are solved in terms of the Lambert function. The Landau gauge gluon propagator has been considered in the renormalization group invariant analytic approach (IAA). It is shown that there is a nonperturbative ambiguity in determination of the anomalous dimension function of the gluon field. Several analytic solutions for the propagator at the one-loop order are constructed. Properties of the obtained analytical solutions are discussed.

1 Introduction

One of the most important tasks in quantum chromodynamics (QCD) is to find the momentum dependence of the invariant (running) coupling $\bar{\alpha}_s(Q^2)$. For large Q^2, the perturbative approximations of the coupling are reliable, since the theory is asymptotically free. Outside of the asymptotic region, the perturbative invariant coupling is in fact large. Moreover, it has wrong analytical properties: the unphysical Landau singularities [1, 2] appear at small space-like momenta. As a consequence, renormalization group (RG) improved expressions for perturbation theory (PT) approximations of physical quantities also have incorrect analytical properties. In particular, the electroweak current-current correlation functions parametrized by the running coupling in PT do not obey dispersion relations (DRs).

DRs are fundamental for proving many important results in quantum field theories [3, 4, 5]. There exists a vast theoretical literature for the application of DRs in deep-inelastic lepton-hadron scattering. This problem has been studied in rigorous manner on the basis of Jost-Lehmann-Dyson representation [4]. Other notable examples are the e^+e^- annihilation ratio $R_{e^+e^-}$, and the ratio of hadron to leptonic τ-decay widths R_τ. These time-like quantities can be related to the hadronic two-point correlation functions through DRs. These DRs provide a well-defined method for definition of the running coupling for time-like momentum.

In the infrared region (IR), the investigation encounters great difficulties due to a failure of PT. Several methods developed based on dispersion relation [6, 7], Borel summation and renormalons [7, 8], renormalization scheme (RS) choices [9] and the Schwinger-Dyson

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equations (SDEs) [10]. Common belief is that the unphysical singularities could be eliminated when one adds higher order perturbative and nonperturbative corrections [3, 11].

The widespread supposition is that infrared properties of the exact effective coupling may lead to color confinement in QCD. Most of the suggestions of a confinement mechanism imply the assumption that the invariant coupling has a singularity at $Q^2 = 0$. For instance, in the linear confinement picture, it is assumed that $\bar{\alpha}_s(Q^2) \sim Q^{-2}$, $Q^2 \to 0$. So the Landau pole moves out of the space-like region and causality is restored [12]. It was confirmed that this asymptotic behavior is consistent with the SDEs and Slavnov-Taylor identities of QCD [12, 13, 14].

On the contrary, other authors have argued that the invariant coupling should be finite in the IR region. It is assumed that the coupling "freezes" at low energies due to an infrared-stable fixed point of the theory. Such a possibility has been occurred in the K scheme [15, 16], where the corresponding β-function in the three-loop approximation acquires an infrared fixed point. The similar conclusion has been achieved using Pade approximant methods [17] or the Banks-Zaks expansion [18]. Note that infrared "freezing" can not be in accord with the above mention linear confinement mechanism. However, there is an approach to confinement based on the BRST algebra and the RG methods (the so-called metric confinement) [19, 20]. A linear quark-antiquark potential is not implied within this framework. Instead, an approximately linear potential is supposed [21].

One possible resolution of the "Landau-ghost" problem has been suggested long time ago in Ref. [1]. In Ref.[2], the idea to combine the RG invariance and the Q^2-analyticity (in the context of quantum electrodynamics (QED)) have been proposed. A RG invariant analytic version of PT has been elaborated in this work. Recently, this method has been successfully applied in QCD. It was shown, that the analytic running coupling is stable for the whole interval of momentum, and has an universal infrared limit at $Q^2 = 0$ [22]. The "analyticized" perturbative approximations, as opposed to standard PT, exhibit reduced RS dependence and the results are not sensitive to higher-loop corrections [23]. These features of RG invariant analytic approach (IAA) have been demonstrated in recent calculations of $R_{e^+e^-}$ [24] and R_τ [25]. The calculations have been performed to next-to-next-to-leading order. Other improvement of the analytic approach is that it provides a self-consistent determination of the running coupling in the time-like region [26]. In Refs. [29] the Bjorken and the Gross-Llewellyn Smith sum rules in IAA are considered. The Landau gauge gluon propagator in IAA has been studied in Ref.[33]. For most recent applications of the analytic approach to QCD see the recent review [31].

In Sec.2 we consider some technical aspects of the RG IAA. The two-loop running coupling is solved explicitly in terms of the Lambert W function. This allows us to investigate analytic structure of the coupling in the complex Q^2 plane. The corresponding analytic solution is reconstructed via the dispersion relation. To discuss the RS dependence, we compare the exact expressions for the causal coupling in the two different renormalization schemes. In Sec.3 the structure of the β-function in APT is studied and other approaches to the Landau pole problem are discussed. The special (nonperturbative) model β-function is investigated. In this case, we also obtain the explicit solution for the coupling in terms of the Lambert W function and show that the solution is consistent

with causality. In Sec.4 we apply the analytic approach to the gluon propagator in the Landau gauge. Several ways of restoring analyticity for the propagator are considered. The ambiguity in causal "analyticization procedure" is discussed [28]. Sec.5 contains the concluding remarks.

2 Lambert's W function in IAA

The running coupling of massless QCD satisfies the differential equation [a]

$$Q^2 \frac{d}{dQ^2} \bar{\alpha}_s(Q^2) = \beta(\bar{\alpha}_s(Q^2)), \tag{1}$$

with the initial condition

$$\bar{\alpha}_s(\mu^2) = \alpha_s, \tag{2}$$

here $\alpha_s = \frac{g^2}{4\pi}$, g is the renormalized coupling constant, and μ is the renormalization point. We assume that the non-trivial β function exists with the perturbative asymptotic expansion in powers of α_s

$$\beta(\alpha_s) \approx -\frac{\beta_0}{4\pi}(\alpha_s)^2 - \frac{\beta_1}{(4\pi)^2}(\alpha_s)^3 - \cdots, \tag{3}$$

the first two coefficients of the formal power series are independent of the chosen renormalization conditions. Their values are

$$\beta_0 = \left(11 - \frac{2}{3}N_f\right), \quad \beta_1 = \left(102 - \frac{38}{3}N_f\right), \tag{4}$$

with N_f being the number of quark flavors. The solution of Eq. (1), which satisfies the initial condition (2), has the form [36]

$$\ln\left(\frac{Q^2}{\Lambda^2}\right) = \frac{4\pi}{\beta_0 \bar{\alpha}_s(Q^2)} - \frac{\beta_1}{\beta_0^2} \ln\left(1 + \frac{4\pi\beta_0}{\beta_1 \bar{\alpha}_s(Q^2)}\right) + \psi(\bar{\alpha}_s(Q^2)), \tag{5}$$

where

$$\psi(\alpha_s) = \int_0^{\alpha_s} \left(\frac{1}{\beta(x)} - \frac{1}{\beta^{(2)}(x)}\right) dx,$$

and $\beta^{(2)}(x)$ denotes the two-loop β-function. The QCD scale parameter Λ here is different from the conventional one $\Lambda_{\overline{MS}}$, so that $\Lambda = (b)^{(\frac{-b}{2})}\Lambda_{\overline{MS}}$ with $b = \frac{\beta_1}{\beta_0^2}$ [36]. To obtain explicit expressions for $\bar{\alpha}_s(Q^2)$ as a function of Q^2, beyond the one-loop order, one must solve the transcendental equation (5).

[a] We use the notation $Q^2 = -q^2$, $Q^2 > 0$ corresponds to a spacelike momentum transfer.

Let $\bar{\alpha}^{(n)}(Q^2)$ be nth-order perturbative approximation to $\bar{\alpha}_s(Q^2)$ determined implicitly by Eq. (5). Then the corresponding analytic running coupling can be defined via the Källen-Lehmann integral [2, 22]

$$\bar{\alpha}_{an}^{(n)}(Q^2) = \frac{1}{\pi}\int_0^\infty \frac{\rho^{(n)}(\sigma)}{(\sigma + Q^2 - i0)}d\sigma, \tag{6}$$

the spectral function $\rho^{(n)}(\sigma)$ is the discontinuity of the "initial" expression $\bar{\alpha}^{(n)}(Q^2)$ along the negative Q^2 axis: $\rho^{(n)}(\sigma) = Im\bar{\alpha}^{(n)}(-\sigma - i0)$. The remarkable result is that the limiting value $\bar{\alpha}_{an}(0)$ is universal and determined only by one-loop calculation [22]:

$$\bar{\alpha}_{an}^{(n)}(0) = \bar{\alpha}_{an}^{(1)}(0) = \frac{4\pi}{\beta_0}.$$

By using Cauchy's theorem Eq. (6) can be rewritten as follows

$$\bar{\alpha}_{an}^{(n)}(Q^2) = \bar{\alpha}^{(n)}(Q^2) + \theta^{(n)}(Q^2). \tag{7}$$

Here the "nonperturbative" term $\theta^{(n)}$ (which compensates the unphysical contributions of $\bar{\alpha}^{(n)}$) comes from the ghost cut

$$\theta^{(n)}(Q^2) = \frac{1}{\pi}\int_{k_L^2}^0 \frac{\rho_\theta^{(n)}(\sigma)}{(\sigma + Q^2 - i0)}d\sigma, \tag{8}$$

where $k_L^2 < 0$, to first and second orders $k_L^2 = -\Lambda^2$.

Let us now consider the two-loop running coupling in more detail. It is more convenient to introduce the quantity

$$a(x) = \tfrac{\beta_0}{4\pi}\bar{\alpha}(Q^2), \quad \text{where} \quad x = \tfrac{Q^2}{\Lambda^2}. \tag{9}$$

To second order Eq. (5) reads

$$\frac{1}{a^{(2)}(x)} - b\ln\left(1 + \frac{1}{b}\frac{1}{a^{(2)}(x)}\right) = \ln x, \tag{10}$$

where we have denoted $b = \frac{\beta_1}{{\beta_0}^2}$ ($b = \frac{64}{81}$ for $N_f = 3$). In what follows we shall consider the phenomenologically interesting case when $N_f \leq 8.05$ ($0 \leq b < 1$). Then the two-loop running coupling has the ghost singularity on the positive x-axis. The transcendental equation (10) can be solved by the iteration method. The solution given by one iteration is [b]

$$a_{it}^{(2)}(x) = \frac{1}{\ln x + b\ln\left(1 + \frac{1}{b}\ln x\right)}. \tag{11}$$

[b] Historically, for the first time the 2-loop iterative solution has been proposed in Ref. [32] in the context of QED.

Consider function (11) in the whole complex x-plane. It has the Landau pole at $x = 1$, the logarithmic branch point at $x = exp(-b)$ and standard branch point at $x = 0$. The corresponding analytic coupling is given by the Källen-Lehmann integral (see Refs.[22])

$$a^{(2)}_{it.an}(x) = \frac{1}{\pi}\int_0^\infty \frac{\bar{\rho}^{(2)}_{it}(s)}{s+x}ds = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^t}{e^t+x}\tilde{\rho}^{(2)}_{it}(t)dt, \tag{12}$$

where $\tilde{\rho}^{(2)}_{it}(t) \equiv \bar{\rho}^{(2)}_{it}(e^t) \equiv \rho^{(2)}_{it}(e^t\Lambda^2) = \frac{L_1}{(L_1^2+L_2^2)}$ with

$$L_1 = \pi + b\arccos(\frac{b+t}{r}), \quad L_2 = t + b\ln(\frac{r}{b}), \quad r = \sqrt{(b+t)^2+\pi^2}.$$

One can investigate Eq.(10) using the lagrange inversion formula. Then one finds that the solution has second order branch point at $x = 1$: $a^{(2)}(x) \sim \frac{1}{\sqrt{2b(x-1)}}$. Hence, approximation (11) violates the analytical properties of $a^{(2)}(x)$ near the point x=1. In fact, the transcendental Eq. (10) is exactly solvable [34, 35, 37]. The solution is

$$a^{(2)}(x) = -\frac{1}{b}\frac{1}{1+\omega(x)} \;:\;\; \omega(x) = W(\zeta), \tag{13}$$

where

$$\zeta = -\frac{1}{e}x^{-\frac{1}{b}} = exp\left(-\frac{\ln(x)}{b} - 1 + i\pi\right), \tag{14}$$

and $W(\zeta)$ denotes the Lambert W function [38]. This is the multivalued inverse of

$$\zeta = W(\zeta)\exp W(\zeta).$$

The branches of W are denoted $W_k(\zeta), k = 0, \pm1, \ldots$. A detailed review of properties and applications of this special function can be found in [38]. We shall define branches of W following this work. Then the branch cuts are chosen as $\{\zeta : -\infty < \zeta \leq -\frac{1}{e}\}$ and $\{\zeta : -\infty < \zeta \leq 0\}$ [c]. The branch $W_{-1}(\zeta)$ satisfies $W_{-1}(\zeta) \leq -1$ for $-e^{-1} \leq \zeta \leq 0$. Now our task is to choose the suitable branches of W in Eq. (13). The solution $a^{(2)}(x)$ is supposed to have the following properties:
i. It would be an analytic function in the cut x-plane with cuts $\{x : -\infty < x < 0\}$ and $\{x : -\infty < x \leq 1\}$.
ii. It is real and positive for a real positive $x > 1$.
iii. $a^{(2)}(x) \sim 1/\ln x$ if $x \to \infty$ along all directions in the complex x-plane.
It is obvious, that for real positive $x > 1$ the relevant branch is $\omega(x) = W_{-1}(\zeta)$. We look for the analytical continuation of $\omega(x)$ to the whole complex x plane. To perform the analytical continuation, one has to take into account the ranges of the branches of $w = W(\zeta)$ [38]. Let us consider the cases $0 < N_f < 6.19$ $(0.5 \leq b < 1)$ and $6.19 \leq N_f \leq 8$ $(0 < b < 0.5)$ separately:

[c]These notations and definitions of branches of W are different from that used in Refs.[34]

1. $0 \le N_f < 6.19$. Evidently, on the upper half-plane ($Im(x) > 0$) the solution is determined by $W_{-1}(\zeta)$. On the lower half-plane the branch should be changed. Indeed, we have crossed the cut in the complex ζ-plane. Accordingly, the replacement $\arg(\zeta) \Rightarrow \arg(\zeta) - 2\pi$ is implied. We conclude that here the relevant branch is $W_1(\zeta)$. With this choice, the solution is continuous on the line $1 < x < \infty$. Thus we obtain

$$\omega(x) = \begin{cases} W_{-1}(|\zeta| e^{i\varphi_1}): & \varphi_1 = \pi - \frac{1}{b}\arg(x) \quad \text{if } 0 < \arg(x) \le \pi. \\ W_1(|\zeta| e^{i\varphi_2}): & \varphi_2 = -\pi - \frac{1}{b}\arg(x) \quad \text{if } -\pi < \arg(x) \le 0. \end{cases} \tag{15}$$

Here $|\zeta| = \frac{1}{e}|x|^{-\frac{1}{b}}$.

2. $6.19 \le N_f \le 8$. In this case, we separate the upper half-plane, ($\arg(x) \in (0, \pi)$), into sectors

$$2(n-1)b\pi < \arg(x) \le \min(2nb\pi, \pi), \quad 1 \le n \le n_{max}, \quad \frac{1}{2b} \le n_{max} < \frac{1}{2b} + 1,$$

(for $N_f = 8$, $n_{max} = 25$). An image of a sector is a sheet of the Riemannian surface of ζ, which maps onto the range of a branch of $\omega(x) = W(\zeta)$. The analytical continuation to the n-th sector can be written as

$$\omega(x) = W_{-n}(|\zeta| e^{i\varphi_{-n}}), \quad \text{where } \varphi_{-n} = (2n-1)\pi - \frac{1}{b}\arg(x). \tag{16}$$

To obtain the solution on the lower half-plane, ($\arg(x) \in (-\pi, 0)$), one must use the formula $W_n(\overline{\zeta}) = \overline{W_{-n}(\zeta)}$ [38].

Now we can reconstruct the corresponding analytic coupling $a^{(2)}_{an}(x)$. In the case

Table 1: The Q^2 dependence of the analytic running couplings $\alpha^{(2)}_{an}(Q^2)$, $\alpha^{(2)}_{it.an}(Q^2)$ and $\alpha^{spec}_{an}(Q^2)$ for $N_f = 3$. Here, we have used the reference value of $\bar{\alpha}_s({M_\tau}^2) = 0.36$ for $M_\tau = 1.777 GeV$.

$Q^2\ GeV^2$	$\alpha^{(2)}_{an}(Q^2)$	$\alpha^{(2)}_{it.an}(Q^2)$	$\alpha^{spec}_{an}(Q^2)$		$Q^2\ GeV^2$	$\alpha^{(2)}_{an}(Q^2)$	$\alpha^{(2)}_{it.an}(Q^2)$	$\alpha^{spec}_{an}(Q^2)$
0	1.396	1.396	1.396		5	0.330	0.330	0.330
0.001	0.985	0.969	0.980		6	0.319	0.319	0.319
0.01	0.846	0.831	0.838		7	0.310	0.310	0.310
0.05	0.716	0.706	0.710		8	0.302	0.302	0.303
0.1	0.654	0.647	0.649		9	0.296	0.296	0.296
0.5	0.507	0.505	0.505		10	0.290	0.290	0.291
1	0.447	0.446	0.446		20	0.256	0.257	0.257
2	0.393	0.392	0.392		30	0.239	0.239	0.240
3	0.364	0.364	0.364		40	0.228	0.228	0.229
4	0.344	0.344	0.344		50	0.220	0.220	0.221

$0 \le N_f \le 6$, with Eqs. (6), (13) and (15), we obtain

$$a^{(2)}_{an}(x) = \frac{1}{\pi}\int_0^\infty \frac{\bar{\rho}^{(2)}(s)}{s+x} ds = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^t}{(e^t + x)} \tilde{\rho}^{(2)}(t) dt, \tag{17}$$

Table 2: Numerical results for Λ and $A(2GeV)$ with exact and iterative solutions.

$\bar{\alpha}_s(M_\tau{}^2)$	0.34	0.36	0.38
$\Lambda_{it}^{(2)}$(GeV)	0.606	0.706	0.815
$\Lambda^{(2)}$(GeV)	0.665	0.772	0.889
Λ_{spec}(GeV)	0.644	0.750	0.865
$A_{it}^{(2)}(2GeV)$	0.476	0.499	0.522
$A^{(2)}(2GeV)$	0.479	0.502	0.525
$A_{spec}(2GeV)$	0.477	0.500	0.523

where $\tilde{\rho}^{(2)}(t) \equiv \bar{\rho}^{(2)}(e^t) \equiv \rho^{(2)}(e^t\Lambda^2)$ and

$$\tilde{\rho}^{(2)}(t) = -\tfrac{1}{b} Im\left(\tfrac{1}{1+W_1(\zeta_1(t))}\right), \quad \zeta_1(t) = \exp\left(-\tfrac{t}{b} - 1 + i(\tfrac{1}{b} - 1)\pi\right). \tag{18}$$

Equivalently, using the Cauchy formula we can rewrite Eq. (17), in the form

$$a_{an}^{(2)}(x) = a^{(2)}(x) - \frac{1}{\pi}\int_0^1 \frac{1}{s-x} Im\{a^{(2)}(s+i0)\}ds.$$

In particular, for $0 \le x \le 1$, we have

$$a_{an}^{(2)}(x) = Re\{a^{(2)}(x+i0)\} - PV\frac{1}{\pi}\int_0^1 \frac{1}{s-x} Im\{a^{(2)}(s+i0)\}ds,$$

where PV denotes the principal value of the integral, and

$$a^{(2)}(x+i0) = -\frac{1}{b}\frac{1}{1+W_{-1}\left(-\frac{1}{e}x^{-\frac{1}{b}}\right)}.$$

Numerical results for the exact two-loop coupling (17) as well as for the iterative solution (12) are summarized in the Table 1. The relative error for the iterative solution (12), is less then 1.8 % for the considered interval. This gives 8% error in the value of the QCD scale parameter $\Lambda^{(2)}$ for $N_f = 3$ (see Table 2). Following Refs.[22] we may use the average

$$A(Q) = \frac{1}{Q}\int_0^Q \bar{\alpha}_s(\mu^2)d\mu$$

for comparison. For $Q = 2$ GeV, we find that iterative solution (12) gives answer for $A(Q)$ to much better than 0.5% accuracy. The reference values of $\alpha_{an}(M_\tau{}^2)$ are taken as: $\alpha_{an}(M_\tau{}^2) = 0.36 \pm 0.02$ with $M_\tau = 1.777$ GeV. The instructive example is given by the

β-function of the special RS

$$\beta(\alpha_s) = -\frac{\beta_0}{4\pi}\frac{\alpha_s{}^2}{1-\frac{\beta_1}{4\pi\beta_0}\alpha_s}, \tag{19}$$

in this scheme the inverse β-function contains only two terms (for application of this RS see.[39]). Note that expression (19) is beyond the formal framework of PT where only finite order polynomials in the coupling are allowed [9]. The RG equation (1) with (19) yields the implicit solution for the coupling

$$x = (ba(x))^b \exp(\tfrac{1}{a(x)}) : \quad x = \tfrac{Q^2}{\Lambda^2}, \quad b = \tfrac{\beta_1}{\beta_0{}^2}, \tag{20}$$

here Λ we define following Ref.[36]. An inversion of (20), can be written in terms of the Lambert W function

$$a(x) = -\tfrac{1}{b\omega_1(x)} : \quad \omega_1(x) = W(z), \quad z = -x^{-\frac{1}{b}}. \tag{21}$$

For $x > e^b$, ($b > 0$, for $N_f \leq 8$) the physical branch is $W_{-1}(z)$. This branch is real and yields the correct ultraviolet behavior for the $a(x)$. We see that $a(x)$ has the branch point (the unphysical singularity) at $x = e^b$, and is finite at this point

$$a(x) \sim \tfrac{1}{b} - \tfrac{1}{b}\sqrt{\tfrac{2(x-e^b)}{be^b}}, \quad x \sim e^b.$$

Let us now perform the analytic continuation in the complex x-plane. For $0 \leq N_f \leq 6$ we obtain

$$\omega_1(x) = \begin{cases} W_{-1}(|z|e^{i\varphi_+}) : & \varphi_+ = \pi - \frac{1}{b}\arg(x) \quad \text{if } 0 < \arg(x) \leq \pi. \\ W_1(|z|e^{i\varphi_-}) : & \varphi_- = -\pi - \frac{1}{b}\arg(x) \quad \text{if } -\pi < \arg(x) \leq 0. \end{cases} \tag{22}$$

The corresponding causal coupling is defined via the DR

$$\tilde{a}_{an}(x) = \frac{1}{\pi}\int_0^\infty \frac{\bar{\rho}_1(s)}{s+x}ds = \frac{1}{\pi}\int_{-\infty}^\infty \frac{e^t}{(e^t+x)}\tilde{\rho}_1(t)dt, \tag{23}$$

where

$$\tilde{\rho}_1(t) = \bar{\rho}_1(e^t) = -\frac{1}{b} Im\left(\frac{1}{W_1\left(e^{-\frac{t}{b}+i\pi(\frac{1}{b}-1)}\right)}\right)$$

Let us compare the exact results (17) and (23) (note that the exact two loop expression (17) is the coupling of the 't Hooft scheme). Numerical results are summarized in the Table 1. The difference between the solutions in the infrared region is less then 1%.

There are several reasons to believe that the application of the Lambert function may be useful from practical point of view:

1) Of course, the transcendental equation (5) can be solved in the complex domain using numerical methods [24, 25]. However, this implies a preliminary analytical investigation: It is necessary to determine the analyticity structure of the implicit function (the relevant branch should be chosen). The application of the Lambert W function facilitates this task.

2) The explicit inverted 2- and 3-loop solutions of RG equation may also be useful in studies of renormalon properties of gauge theories [51].

Note added. While preparing this manuscript I became aware of the paper [37] which has some overlap with the present work. The authors of [37] have obtained explicit expressions for the coupling (at the 2-loop and 3-loop orders) in terms of the Lambert W function.

3 The β-function in the IAA

It is instructive to consider the β-function. From Eqs. (1), (2) and (6) we have

$$\beta_{an}^{(n)}(\alpha_s) = -\frac{1}{\pi}\int_0^\infty \frac{\mu^2}{(\sigma+\mu^2)^2}\rho^{(n)}(\sigma)d\sigma = -\frac{1}{\pi}\int_0^\infty \frac{\bar{\rho}^{(n)}(s)}{y(s+\frac{1}{y})^2}ds, \tag{24}$$

where $\bar{\rho}^{(n)}(s) \equiv \rho^{(n)}(\sigma)$, $s = \frac{\sigma}{\Lambda^2}$, and y denotes the dimensionless "nonperturbative" variable

$$y = exp(-\phi) = \frac{\Lambda^2}{\mu^2}, \tag{25}$$

for $\alpha_s \to 0$, $y \sim \exp(-\frac{4\pi}{\beta_0\alpha_s})$. The spectral representation (24) defines $\beta(\alpha_s) \equiv \hat{\beta}(y)$ as an analytic function in the cut-y plane. The cut is along the negative y-axis. Another useful representation for the β-function follows from Eqs. (2) and (7)

$$\beta_{an}^{(n)}(\alpha_s) = \frac{\partial\alpha_s}{\partial ln\mu^2} = \beta_{pt}^{(n)}(\bar{\alpha}^{(n)}(\mu^2)) + \bar{\theta}^{(n)}(\mu^2) = \beta_{pt}^{(n)}(\alpha_s - \theta^{(n)}(\mu^2)) + \bar{\theta}^{(n)}(\mu^2), \tag{26}$$

where $\beta_{pt}^{(n)}$ denotes the nth-order perturbative approximation for the β-function

$$\beta_{pt}^{(n)}(\alpha) = -\sum_{k=0}^{n-1}\frac{\beta_k}{(4\pi)^{k+1}}\alpha^{k+2},$$

and $\bar{\theta}^{(n)}(\mu^2) = \mu^2\frac{\partial\theta^{(n)}(\mu^2)}{\partial\mu^2}$. Eq. (26) can be rewritten as follows

$$\beta_{an}(\alpha_s) = \beta_{pt}(\alpha_s) + \beta_{np}(\alpha_s, y). \tag{27}$$

Here, the nonperturbative piece, β_{np}, denotes the generalized power expansion for α_s

$$\beta_{np}^{(n)}(\alpha_s, y) = \sum_{k=0}^{n} B_k^{(n)}(y)(\alpha_s)^k, \tag{28}$$

where

$$B_0 = \bar{\theta}, \quad B_1 = -\frac{\partial\beta_{pt}(-\theta)}{\partial\theta}, \quad B_k = \frac{(-1)^k}{k!}\frac{\partial^k\beta_{pt}(-\theta)}{\partial\theta^k} + \frac{\beta_{k-2}}{(4\pi)^{k+1}} \quad \text{for} \quad k \geq 2. \tag{29}$$

Here, for convenience, we use condensed notations suppressing the superscript (n) and arguments to the expressions above ($\theta \equiv \theta^{(n)}(\mu^2)$ ets.). From the spectral representation (8) and Eq. (29) we see that the coefficients $B_k(y)$ are analytical functions of y in the cut y-plane. The cut is along the real positive y-axis $y \geq \frac{1}{k_L} > 0$. In particular, the coefficients are regular functions in the neighborhood of $y = 0$. For $\alpha_s \to 0$, the coefficients vanish exponentially $B_k(y) \sim y \sim \exp(-\frac{4\pi}{\beta_0\alpha_s})$.

A detailed study of the one-loop β-function can be found in Ref. [27]. It was shown that the one-loop β-function obeys the symmetry property $\beta(\alpha_s) = \beta(\frac{4\pi}{\beta_0} - \alpha_s)$, and it

has the second order zero at $\alpha_s = \frac{4\pi}{\beta_0}$. To second order the symmetry property for the β-function is not occurred.

Notice that in the RG IAA the nonperturbative $exp(-\frac{1}{a_s})$-type terms play an essential role for restoring of analyticity. The reason is that one starts with a finite order perturbative approximation to the β-function. Another point of view has been advocated in Refs. [9]. It was shown that a perturbative series for the β-function, in the specific class of schemes, can be summed. So that the resulting running coupling has causal structure. The nonperturbative terms are not added, but the freedom of choosing a RS for an infinite series was used instead [9]. The connection between that approach and IAA is still unclear. Indeed, they offer two different causal approximations for the running coupling.

To illustrate we consider the model β-function introduced in Ref. [40]

$$\bar{\beta}(h) = -\frac{\beta_0 h^2}{1+\beta_0 h}: \quad h = \frac{\alpha_s}{4\pi}, \quad \bar{\beta}(h) = 4\pi\beta(\alpha_s), \tag{30}$$

which yields a linearly rising potential and asymptotic freedom. The RG equation with (30) can be solved, the solution is

$$\bar{h}(\frac{Q^2}{\Lambda^2}) = \frac{1}{\beta_0 W_0\left(\frac{Q^2}{\beta_0\Lambda^2}\right)}: \quad \Lambda^2 = h\mu^2 exp(-\frac{1}{\beta_0 h}), \tag{31}$$

here $W_0(z)$ is principal branch of $W(z)$ [38]. Quite remarkably, (31) satisfies the cut-plane analyticity. Indeed, $W_0(z)$ is analytic at $z = 0$ ($W_0(z) = z - z^2...$) and its branch cut is $\{z : -\infty < z < -1/e\}$. A generalization of this model has been studied in Ref.[14].

Generally, the running coupling is a gauge- and scheme-dependent quantity (it is nonphysical object in this sense [9]). Presumably, more useful quantities, in the infrared region, are the scheme-independent effective charge (the combination of a vertex and propagators) [41]) or the running couplings of physical schemes [15, 16, 42]. Most satisfactory approach can be based on the notion of scheme- and gauge-independent universal charge (the analogous of the QED effective charge) [6]. Recently, in Ref. [43] the universal one-loop QCD coupling has been constructed using the pinch technique.

It is clear that IAA does not exhaust all nonperturbative effects of the invariant coupling because its origin is PT. In Refs. [6, 7, 44] more general frameworks are considered. It is assumed that a nonperturbative universal QCD coupling exists which satisfies the DR.

We can write the nonperturbative universal coupling as a decomposition

$$\alpha_{eff}(Q^2) = \alpha_{eff}^{IR}(Q^2) + \alpha_{eff}^{UV}(Q^2).$$

The infrared and ultraviolet pieces of the coupling satisfy the DRs

$$\alpha_{eff}^{IR}(Q^2) = \frac{1}{\pi}\int_0^{K^2(\alpha_s,\mu)} \frac{\rho^{IR}(\sigma)}{\sigma+Q^2-i0}d\sigma, \quad \alpha_{eff}^{UV}(Q^2) = \frac{1}{\pi}\int_{K^2(\alpha_s,\mu)}^{\infty} \frac{\rho^{UV}(\sigma)}{\sigma+Q^2-i0}d\sigma. \tag{32}$$

The momentum $K(\alpha_s, \mu)$ divides the infrared and ultraviolet regions. It may be chosen in a RG-invariant fashion. In Refs. [48] it was shown that this nonperturbative scale actually

exists in QCD if $N_f \leq 9$. In the weak coupling limit the scale vanishes exponentially, $K(\alpha_s,\mu) \sim \Lambda \sim e^{-\frac{4\pi}{\beta_0\alpha_s}}$ as $\alpha_s \to 0$. The APT approach corresponds to the simplest possibility: it is assumed that $K=0$, hence $\alpha^{ir}_{eff}(Q^2)$ is ignored. In addition, the weight function $\rho^{UV}(\sigma)$ is approximated by PT.

One can, in principle, examine the consistency of IAA with a truncated set of SDEs. The authors of Refs. [44] have given reasons to believe that $\alpha^{IR}_{eff}(Q^2)$ cannot be zero, and an infrared regular QCD effective charge cannot be consistent with the SDE for the gluon propagator. Ghost-free axial gauge has been chosen in this work. On the contrary, the authors of Ref.[45] have found that the infrared-regular nonperturbative effective charge is consistent with the truncated SDEs in the Landau gauge. However, it was shown that the corresponding gluon propagator has not causal structure. We see that situation is not completely clear.

4 The gluon propagator in IAA

In general the gluon propagator is not observable. But, this fact does not imply that it does not contain physics. One can relate the propagator to gauge invariant quantities, for example the Wilson loop [46], the gluon condensate [44] or the gluon distribution function of hadrons [47]. Although the propagator is known to be a gauge variant quantity it contains an important information about the infrared region [10, 12, 13, 14].

It has been proved that the propagators in QCD obey the DRs [48]. In the standard RG improved PT the DRs are violated [49]. Therefore it is desirable to develop the analytical approach for calculating of the propagators. It was pointed out in Ref. [28] that there is an ambiguity due to "noncommutativity" of "analyticization" with some elements of the RG algorithm. For this reason, in the case of Green's functions and observables, several different versions of "analyticization" are possible. To illustrate we shall consider the gluon propagator in the Landau gauge

$$D_{\mu\nu}(Q) = (g_{\mu\nu} + \tfrac{Q_\mu Q_\nu}{Q^2})D(Q^2): \quad D(Q^2) = \frac{d(\frac{Q^2}{\mu^2},\alpha_s)}{Q^2}, \quad d(1,g^2) = 1. \tag{33}$$

We have assumed that quarks are massless and normalized the propagator at the Euclidean point $Q^2 = \mu^2$. In the Landau gauge, invariance under the RG leads

$$d\left(\frac{Q^2}{\mu^2},\alpha_s\right) = \exp\left(\int_{\alpha_s}^{\bar{\alpha}_s(Q^2)} \frac{\gamma_v(x)}{\beta(x)}dx\right). \tag{34}$$

The formal power series for the anomalous dimension function γ_v (in this gauge) is given by

$$\gamma_v(\alpha_s) = -(\tfrac{\gamma_0}{4\pi}\alpha_s + \tfrac{\gamma_1}{16\pi^2}{\alpha_s}^2 + \ldots), \quad \gamma_0 = \tfrac{1}{2}(13 - \tfrac{4}{3}N_f). \tag{35}$$

A detailed investigation of the gluon propagator in the framework of analyticity and asymptotic freedom has been undertaken in Refs.[48]. It has been shown that the propa-

gator amplitude satisfies an unsubtracted DR

$$D(Q^2,\alpha_s,\mu)=\frac{1}{\pi}\int\limits_0^\infty\frac{\rho_v(\sigma,\alpha_s,\mu)}{\sigma+Q^2-i0}d\sigma. \tag{36}$$

For a limited number of flavors $N_f\le 9$, in the Landau gauge, the weight function ρ_v obeys the superconvergence relation

$$\int\limits_0^\infty\rho_v(\sigma,\alpha_s,\mu)d\sigma=0. \tag{37}$$

This relation is considered as a sufficient condition for color confinement in the approach to a confinement based on the BRST algebra and the RG [19, 20]. The consequence of (37) is that there is a renormalization invariant point $K^2(\alpha_s,\mu)$ such that ρ_v is a negative measure for $\sigma\ge K^2$ [48].

Let us consider few different scenarios of restoring analyticity for the gluon propagator:

(a) The Analytic Perturbation Theory (APT) possibility. This specific recipe for "analyticization" of an observable has been introduced and elaborated in Refs.[25, 29, 24]. Here, instead of the power perturbation series, an amplitude is presented in a form of an asymptotic expansion of a more general form, the expansion over an asymptotic set of functions $[a^n(x)]_{an}$, the "n-th power of $a(x)$ analyticized as a whole" [28]. In the APT approach, the drastic reduction of loop and RS sensitivity for several observables has been found.

In this case the starting point is formula (34). In the one-loop approximation (34) yields

$$D^{(1)}(\frac{Q^2}{\mu^2},\alpha_s)=\frac{1}{Q^2}\left(\frac{\bar{\alpha}^{(1)}(Q^2)}{\alpha_s}\right)^{\frac{\gamma_0}{\beta_0}}=\frac{1}{Q^2}\left(\frac{4\pi}{\alpha_s\beta_0\ln\left(\frac{Q^2}{\Lambda^2}\right)}\right)^{\frac{\gamma_0}{\beta_0}}. \tag{38}$$

Since the ratio $\frac{\gamma_0}{\beta_0}$ is fractional number, expression (38) has the ghost cut $0<Q^2<\Lambda^2$. The discontinuity $\rho_v^{(1)}$ of $D^{(1)}$ along the negative Q^2 axis can be written as [49]

$$\rho_v^{(1)}(\sigma)=\frac{\bar{\rho}_v^{(1)}(s)}{\Lambda^2};\quad \bar{\rho}_v^{(1)}(s)=-c_v(\alpha_s)\frac{1}{s}(R(s))^{\frac{\gamma_0}{\beta_0}}\sin\left[\frac{\gamma_0}{\beta_0}\Phi(s)\right], \tag{39}$$

where $s=\frac{\sigma}{\Lambda^2}$, $c_v=\left(\frac{4\pi}{\beta_0\alpha_s}\right)^{\frac{\gamma_0}{\beta_0}}$, $R(s)=(\ln^2 s+\pi^2)^{-\frac{1}{2}}$ and

$$\Phi(s)=\begin{cases}\arcsin(\pi R(s)) & \text{if } s>1\\ \pi-\arcsin(\pi R(s)) & \text{if } 0<s<1.\end{cases}$$

To construct the corresponding analytic expression one has to substitute (39) in formula (36). Note that the weight function (39) has a nonintegrable singularity at $\sigma=0$. So that the spectral representation (36) with the weight function (39) diverges. Nevertheless,

in sense of the distribution theory this problem may be solved [d]. Using the method of Refs. [48] we find the following subtracted DR for the "analyticized" amplitude

$$D^{(1)}_{an1}(Q^2) = \frac{c_1}{Q^2} - \frac{1}{Q^2\pi}\int\limits_0^1 \frac{s\bar{\rho}^{(1)}_v(s)ds}{s+\frac{Q^2}{\Lambda^2}} + \frac{1}{\Lambda^2\pi}\int\limits_1^\infty \frac{\bar{\rho}^{(1)}_v(s)ds}{s+\frac{Q^2}{\Lambda^2}}, \tag{40}$$

where

$$c_1 = \tfrac{1}{\pi}\int\limits_0^1 \bar{\rho}^{(1)}_{v.reg}(s)ds; \quad \bar{\rho}^{(1)}_{v.reg} = \bar{\rho}^{(1)}_v - \bar{\rho}^{(1)}_{v.sing}; \quad \bar{\rho}^{(1)}_{v.sing}(s) = -\frac{c_v(\alpha_s)}{s|\ln s|^p}\sin(p\pi), \tag{41}$$

with $p = \frac{\gamma_0}{\beta_0} = \frac{39-4N_f}{2(33-2N_f)}$ ($0 < p < 1$ if $N_f \le 9$), and $\bar{\rho}^{(1)}_v(s)$ is given by (39). Instead of the superconvergence relation (37) now we have the following relation [49]

$$\int\limits_0^1 \bar{\rho}^{(1)}_{v.reg}(s)ds + \int\limits_1^\infty \bar{\rho}^{(1)}_v(s)ds = 0. \tag{42}$$

However, there is another way to handle the above considered infrared problem. The point is, that for $N_f \le 9$ one can write an unsubtracted DR for the dimensionless structure d, $d = Q^2 D(Q^2)$ [48]. This allows us to construct the analytic expression for d. The corresponding "analyticized" amplitude D_{an} is then determined by

$$D^{(1)}_{an1}(Q^2) = \frac{d^{(1)}_{an}(\tilde{Q}^2)}{Q^2} = \frac{1}{Q^2\pi}\int\limits_0^\infty \frac{\bar{\rho}^{(1)}_{v1}(s)ds}{s+\tilde{Q}^2} \tag{43}$$

where $\tilde{Q}^2 = \frac{Q^2}{\Lambda^2}$, $\bar{\rho}^{(1)}_{v1}(s) = -s\bar{\rho}^{(1)}_v(s) > 0$ with $\bar{\rho}^{(1)}_v(s)$ given by Eq.(39). One can verify that expression (43) has the correct "abelian limit". Indeed, we have

$$\lim_{\gamma_0\to\beta_0} a_s d^{(1)}_{an}(\tilde{Q}^2) = \frac{1}{\pi}\int\limits_0^\infty \frac{ds}{s+\tilde{Q}^2}\left(\frac{\pi}{\ln^2 s+\pi^2}\right) = a^{(1)}_{an}(\tilde{Q}^2) \equiv \left(\frac{1}{\ln\tilde{Q}^2} + \frac{1}{1-\tilde{Q}^2}\right).$$

Using the relation (42) we verify that the analytic expression (43) follows from the representation (40).

Note that (43) does not possess a pole at $Q^2 = 0$ and it has a more strong singularity then the corresponding amplitude for the free field. By direct calculation we find

$$D^{(1)}_{an1}(Q^2) \sim \frac{c_v}{\pi(1-p)}(|\ln(\frac{Q^2}{\Lambda^2})|)^{1-p}\frac{1}{Q^2} \quad \text{for} \quad Q^2 \to 0.$$

More serious difficulty is that there is not proper connection between the analytic propagator (40) and the analytic running coupling as it follows from basic RG relations [28]. Indeed, the product of a vertex and appropriate powers of propagators should form an invariant charge.

[d] The infrared behavior of spectral representations has been studied in Refs. [48].

(b) One can substitute the explicit expressions $a_{an}(Q^2)$ in formula (34). The simple possibility is to use the perturbative expression (35) for γ_v. This choice has been proposed in Ref. [50]. Then in the one-loop approximation formula (34) gives the causal expression

$$D^{(1)}_{an2}(Q^2) = \frac{c_{v2}}{Q^2}\left[\frac{\tilde{Q}^2 - 1}{\tilde{Q}^2 \ln \tilde{Q}^2}\right]^{\frac{\gamma_0}{\beta_0}}, \tag{44}$$

where $\tilde{Q} = \frac{Q}{\Lambda}$, and c_{v2} is the normalization constant. As in the case (a), we see that (44) does not satisfy the unsubtracted DR (36). The lack of this solution is that it does not reproduce the "abelian limit" for $\gamma_0 \to \beta_0$.

(c) We may introduce RG improved power asymptotic expansion differing from the usual one by substitution $a_{pert.}(\frac{Q^2}{\Lambda^2}) \Rightarrow a_{an.}(\frac{Q^2}{\Lambda^2})$ only. With this choice, in the one-loop order, we get

$$D^{(1)}_{an3}(Q^2) = \frac{1}{Q^2}\left(\frac{\bar{\alpha}^{(1)}_{an}(Q^2)}{\alpha_s}\right)^{\frac{\gamma_0}{\beta_0}} = \frac{1}{Q^2} c_v \left(\frac{1}{\ln \tilde{Q}^2} + \frac{1}{1 - \tilde{Q}^2}\right)^{\frac{\gamma_0}{\beta_0}}, \tag{45}$$

where $c_v = (\frac{4\pi}{\beta_0 \alpha_s})^{\frac{\gamma_0}{\beta_0}}$ and $\tilde{Q}^2 = \frac{Q^2}{\Lambda^2}$. It is easy to convince that (45) has correct analytical properties. Indeed, $\bar{\alpha}^{(1)}_{an}(Q^2)$ does not vanish in the finite part of the complex Q^2-plane. Moreover, (45) satisfies the unsubtracted DR (see Refs. [33, 34]). For the RG invariant scale $K(g, \mu^2)$, the solution (45) yields trivial value $K = 0$. In a similar way, one can also obtain causal approximations to $D(Q^2)$ at higher orders.

(d) Consider the one-loop gluon self energy $\Pi^{(1)}$,

$$\Pi^{(1)}(\bar{Q}^2, a_s) = \frac{\gamma_0}{\beta_0} a_s \ln \bar{Q}^2, \tag{46}$$

here γ_0 is given by (35), $a_s = \frac{\beta_0}{4\pi}\alpha_s$ and $\bar{Q}^2 = \frac{Q^2}{\mu^2}$. We can introduce the Dyson series in $\Pi^{(1)}$ for the propagator. Summing this series we get

$$d_{Ds}(\bar{Q}^2, a_s) = \frac{1}{1 + \Pi^{(1)}(\bar{Q}^2, a_s)} = -\kappa \ln y \frac{1}{\ln \frac{\bar{Q}^2}{y^\kappa}}, \tag{47}$$

here $\kappa = \frac{\beta_0}{\gamma_0}$ and instead of a_s we have introduced the variable y, $y = e^{-\frac{1}{a_s}}$. The approximation (47) does not satisfy RG invariance and has the ghost pole. We may define the corresponding "analyticized" amplitude [1]

$$d_{Ds.an}(\bar{Q}^2, y) = \frac{1}{1 + \Pi^{(1)}_{mod}(\bar{Q}^2, y)} = C_{Ds}(y)\left(\frac{1}{\ln(\bar{Q}^2 y^{-\kappa})} - \frac{y^\kappa}{\bar{Q}^2 - y^\kappa}\right), \tag{48}$$

in order to preserve the normalization (see (33)) we have introduced in (48) the necessary factor $c_{Ds}(y)$. The effect of this procedure is that the gluon self-energy receives a pure nonperturbative contribution. The modified gluon self-energy function is then given by

$$\Pi^{(1)}_{mod}(\bar{Q}^2, y) = \Pi^{(1)}(\bar{Q}^2, -(\ln y)^{-1}) + \Pi^{(1)}_{nonpert.}(\bar{Q}^2, y), \tag{49}$$

here

$$\Pi^{(1)}_{nonpert.}(\bar{Q}^2, y) = (1 + \Pi^{(1)}(\bar{Q}^2)) \left(-1 + \frac{(\kappa \ln y)^{-1} + G(1)}{(\kappa \ln y)^{-1} + G(\bar{Q}^2)(1 + \Pi^{(1)}(\bar{Q}^2))} \right) \qquad (50)$$

with $\Pi^{(1)}(\bar{Q}^2) \equiv \Pi^{(1)}(\bar{Q}^2, -(\ln y)^{-1})$ and $G(\bar{Q}^2) = y^{\kappa}(\bar{Q}^2 - y^{\kappa})^{-1}$. We see that the expression (49) is free from the ghost singularity. For $N_f \leq 9$ ($\kappa > 0$), in the weak-coupling limit, the nonperturbative part of the self energy vanishes exponentially: $\Pi^{(1)}_{nonpert.}(\bar{Q}^2, y) \sim O(e^{-\frac{\kappa}{a_s}}) \to 0$. The modified one loop amplitude is then given by

$$d^{(1)}_{mod}(\bar{Q}^2, y) = 1 - \Pi^{(1)}_{mod}(\bar{Q}^2, y). \qquad (51)$$

Now, the perturbative definition of y, $y = e^{-\frac{1}{a_s}}$, is not valid. Instead, we shall accept the modified relation $y = \frac{\Lambda^2}{\mu^2} = e^{-\phi(a_s)}$ (see Refs.[2, 27]). Thus, to first order, $\phi(a_s)$ is determined by [27]

$$\frac{1}{\phi(a_s)} + \frac{1}{1 - e^{\phi(a_s)}} = a_s.$$

We can calculate the modified anomalous dimension for the gluon field

$$\gamma^{(1)}_{mod}(a_s) = \bar{\gamma}_{mod}(\phi) = \lim_{\bar{Q}^2 \to 1} \bar{Q}^2 \frac{\partial}{\partial \bar{Q}^2} \ln d^{(1)}_{mod}(\bar{Q}^2, y) = \lim_{\bar{Q}^2 \to 1} \bar{Q}^2 \frac{\partial}{\partial \bar{Q}^2} \ln d_{Ds.an}(\bar{Q}^2, y), \qquad (52)$$

using (48) we find

$$\bar{\gamma}_{mod}(\phi) = \frac{1}{\kappa} \frac{d}{d\phi} \ln \left(\frac{1}{\kappa \phi} - \frac{1}{e^{\kappa \phi} - 1} \right). \qquad (53)$$

Inserting expression (53) in Eq. (34) we obtain the corresponding RG improved expression for the propagator amplitude

$$D_{RG}(Q^2) = \frac{d_{RG}\left(\frac{Q^2}{\Lambda^2}, y\right)}{Q^2} = \frac{C_{RG}(y)}{Q^2} \left(\frac{1}{\kappa \ln\left(\frac{Q^2}{\Lambda^2}\right)} - \frac{1}{\left(\frac{Q^2}{\Lambda^2}\right)^{\kappa} - 1} \right)^{\frac{1}{\kappa}}, \qquad (54)$$

here, the factor $C_{RG}(y)$ is determined by the normalization condition (33). We see that, in the weak-coupling limit $a_s \to 0$ (for $N_f \leq 9$) expression (54) reproduces the standard RG improved solution (38). Furthermore, it has the correct "abelian limit" $a_s d_{RG}(\frac{Q^2}{\Lambda^2}, y) \to a_{an}(\frac{Q^2}{\Lambda^2})$ as $\frac{\beta_0}{\gamma_0} \to 1$.

For $\frac{\gamma_0}{\beta_0} \geq 0.5$, the function (54) satisfies the Källen-Lehmann analyticity. The corresponding flavor condition is $N_f \leq 3$. For $N_f > 3$, the unphysical singularities appear in the first Riemann sheet. This limitation for N_f turns out to be natural. Indeed, the singularities in (54) occurred at $|\frac{k^2}{\Lambda^2}| = 1$ where the number of active quarks are just three. On the other hand, the flavor condition $N_f \leq 3$ seems to be plausible for massless QCD. For this reason we cannot reject the solution (54) using arguments of analyticity [e].

[e] The author owes to D.V. Shirkov for drawing his attention to this peculiarity of the solution (54).

It is interesting to compare the analytic solutions (40), (45) and (54), which have correct "abelian limit". Only (45) and (54) satisfy an unsubtracted DR. The convenient criterion has been formulated in Refs.[44]. This is the principle of minimality for the nonperturbative contributions in perturbative (ultraviolet) region. According to this principle one can easily verify that the solution (54) is preferable. Indeed, it predicts more rapid decrease of the nonperturbative contributions in the ultraviolet region then the solutions (40), (44) and (45). However, we cannot accept a final decision for selecting the solutions. Indeed, the above mention principle is relevant to the full gluon propagator, which may include a pure nonperturbative contributions. In the IAA these contributions are invisible.

5 Conclusion

The RG equation for the QCD two-loop invariant coupling has been solved explicitly. We have expressed the solution in terms of the Lambert W function. This allows us to understand more clearly the analytical structure of the solution in the complex Q^2 plane. The corresponding analytic coupling has been reconstructed via the dispersion relation. It has been demonstrated that the "analyticized" iterative solution (12) is numerically close to the "analyticized" exact one (17).

We have expressed the invariant (running) coupling of the special RS via the Lambert W function. The corresponding analytic invariant coupling is constructed (see (23)). We have shown that the analytic (running) couplings of the two considered schemes are numerically close in the IR region.

The structure of the β-function has been analyzed in IAA. We have solved the RG equation, with nonperturbative model β-function, giving explicit expression for the invariant coupling as a function of the scale also in terms of the Lambert W function (see (31)). We have found that the solution is automatically causal.

The one-loop gluon propagator amplitude of massless QCD is considered in the Landau gauge. The RG and analyticity constraints alone are not sufficient to uniquely determine the analytic solution for the gluon propagator starting from PT. Therefore, several versions of "analyticization" of the gluon propagator are considered. Properties of the obtained analytical solutions for the propagator are discussed.

Finally, we remark that the gluon propagator is central object in the framework based on the SDEs [12, 13, 14]. Here, the analytic perturbative solutions can be used to derive more complete (nonperturbative) approximants to the gluon propagator. For related ideas and applications, see [44].

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